# Authors Note

Please note the original intention of the study was to independently carry out the numerical analyses on both a physical fluid data set and a financial (stock market) data set to independently verify the phenomena discussed in sections 3.1 to 3.3, however finding high resolution data on fluid velocity fluctuation has been deemed too difficult. In fact, the singular source found with potentially the right data had technical issues not allowing me to download a functional version of the file.

Therefore, the independent study could only be carried out on the financial data and compared to the fluid study in the refence materials.

**Fluid and Stock Market Turbulence** *Examples of the Limitation on Quantitative Parallels*

# Introduction

## Nomenclature and Qualitative Parallels

Turbulence is a term that is used in the financial markets, meaning periods of rapid rising and falling of the stock market. The parallels with fluid turbulence are clear when viewed through the lens of airline turbulence, which causes the airplane to rapidly rise and fall, one could conclude that this is the actual origin of the linkage of the two terms in common parlance.

From a qualitative perspective both fluid and financial turbulence can reflect similar characteristics, both have:

* Cyclical periods (intervals with seemingly random lengths) changing between calm and turbulent movement
* Both are a system respond to an outside influence (energy or information respectively)
* If the stream of external influence was to stop, the system would calm to a state where the chaotic behaviour (the turbulence) would calm and then the system would drift to equilibrium
* There isn’t a complete understanding about the equations of motion dictating the exact behaviour of each system

This final point is largely because both phenomena are emergent behaviour from a very large number of multiple smaller particles or market agents responding to an outside influence.

It should be noted that while these qualitative parallels shows similarities, when compared on a quantitative level there are fewer parallels.

## The State of High-Level Turbulence Modelling

Turbulence is a challenging and complex subject that remains “largely unsolved” in various ways including a full mathematical description of its phenomena. Therefore, any equations that describe it are approximations made in ideal/unfeasible situations (usually assuming an infinite or large Reynolds number).

The next section will cover some basic mathematical concepts of turbulence, these the concepts that will then be examined against stock market data in the following section in a recreation of various experimental comparisons between financial and fluid turbulence.

## Methods of Analysis

In the following paper there will be two lenses which the turbulence of fluids and of stock prices will be compared. These are:

* A reflection of the Kolmogorov simplification
* Probability of return
  + The probability that a value will return to its original value after Δt

It will also be shown conclusively that the stock market price in question moves with a leptokurtic distribution (section 4).

# Basic Turbulence Maths

## Reynolds Number

An important equation used to approximate the behaviour of fluid turbulence is the Reynolds number. This number is used to approximate the ratio of inertial to viscous forces in a fluid flow. In an ideal system (here a pipe) the Reynolds number can be calculated by the following equation:

Where:

If this value is below 2000 then the flow is “laminar”, a state where particles tend to travel in smooth, well-ordered layers. Whereas if the value is over 4000 the flow is considered to be turbulent. There is an area between these values called “transitional”, where sections of the flow are turbulent (generally near the contact surfaces) and other sections are still laminar. In this state laminar sections of flow can be converted to turbulent section downstream by random turbulations that upset laminar flow.

## Navier Stokes Equation

While not directly used in the final calculation, a derivative of this equation is used, that being said no discussion of turbulence would be complete without the brief mention of the renowned Navier–Stokes equations:

Where:

These equations are used to describe the motion of fluid in slow-moving (i.e. well below the local speed of sound), incompressible fluids and effectively describe the changes in fluid velocity in both time and space relative to the kinematic viscosity (a function of the fluids viscosity and density) and pressure gradients. Part of what makes these equations difficult is that they consider the gradients in pressure and velocity in 3 spatial and 1 time dimension to give a single rate of change for one dimension in relation to the others.

In fact, the analytical solutions for the Navier–Stokes equation has been proved impossible and numerical solutions for very large Reynolds numbers are currently impossible, which is relevant because that means it can’t be used to calculate, model or simulate turbulent flow.

## Kolmogorov Equation

Kolmogorov showed that in the limit of an infinite Reynolds number, the Navier-Stokes equation could be simplified into a mean square velocity increment:

Which behaves approximately as:

In simple terms the mean square velocity increment is “the average change in velocity squared for a given displacement” and the above equation states that its linearly related to the displacement or the power of 0.66 recurring.

This property can be evaluated if the following relationship can be demonstrated:

This relationship applies for displacement within the inertial range, which are for any scales smaller then where the turbulent behaviour takes place (i.e. the eddies and swirls) but larger than the microscopic scale where viscous forces turn any turbulence into heat energy.

This value can be examined like a second-order moment (a variance) as the distribution of velocity difference can be examined like a distribution.

To calculate this another assumption needs to be made: the Taylor hypothesis, which assumes that that *“advection contributed by turbulent circulations themselves is small and that therefore the advection of a field of turbulence past a fixed point can be taken to be entirely due to the mean flow”* (ametsoc.org, n.d.). Which simply means that for the small scales that we are calculating within we can assume that turbulent features are fixed and moving along the fluid at the average fluid speed, meaning that taking two, sufficiently close, velocity readings can be the equivalent of taking two simultaneous velocity measurements (Cheng, 2017).

# Comparison: Kolmogorov Behaviour or Random Process

This section will see if the two data sets will behave more like the Kolmogorov behaviour explained in section 2.3 or like a random process.

## Fluid Data Available

Figure 1 is the wind velocity 6 metres above the wheat canopy in the Connecticut Agricultural Research Station. This flow was at a high Reynolds number. The time units are cited in multiple places as “arbitrary time units” (Stanley, Stock market dynamics and turbulence: Parallel analysis of fluctuation phenomena, 1997) (Stanley, An Introduction to Enconophysics: Correclations and Complexity in Finance, 1999). This data was sourced from Magenta and Stanley.

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Figure 1: Time evolution of the fluid velocity in fully developed turbulence. (a) Time evolution of the wind velocity recorded in the atmosphere at extremely high Reynolds number; the Taylor microscale Reynolds number is of the order of 1,500. The time units are given in arbitrary units. (b) Velocity differences of the time series given in (a).

## Financial Data Available

Below is the price variation of the S&P500 stock over a period from 2018-10-15 to 2023-06-26, it was sourced from tiingo.com. Each timestep is a minute. This data and any numbers generated from this are the work of myself (Fabio Greenwood).

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Figure 2: (top section) Time evolution of the S&P 500 index, over the period 15/10/2018 to 26/06/2023. (bottom section) Minutely variations of the S&P 500 index in the same period.

Please note that in this paper the variables are named as such:

* Y – the price of a stock
* Z – the change or rate of change in price
* V – the wind velocity
* U – the change in wind velocity

## Reflection of the Kolmogorov Simplification – Results and Discussion

Both data sets were transformed to show the expected standard deviation (change) of fluid velocity or index price, respectively, over different time steps of Δt.

Figure 3 shows the expected standard deviation of fluid velocity over various Δt for the wind data.

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Figure 3: Standard deviation (change) in wind velocity over a given delta v for the wind velocity data demonstrated in Figure 1

Figure 3 shows a super diffusive (meaning faster than random spreading) region in the interval shorter than Δt = 10 a.u. and after shows a log-log relationship between the two values, when this is compared to the power law:

Gives a value for v of 0.33 which is close to the expected value for a fluid of 1/3, confirming the wind demonstrated the expected behaviour outlined by the Kolmogorov equation.

This when compared to the financial data:

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Figure 4: Standard deviation (change) in S&P 500 index price for a given change in time, the data demonstrated in Figure 2

Here the resolution goes to one minute (10^0), no super diffusive region is observed, it is to be assumed that if does exist, it is at a resolution below the 1 min threshold. However in the region showing a log-log relationship (the whole curve) there is a v value of 0.5. This suggests behaviour like a random walk.

# Leptokurtic Distribution

Leptokurtic distributions can be characterised as having higher central peaks and a larger probability of producing extreme values (due to having “fat/heavy” tails). When the change in stock price across 1 minute is placed into a histogram, this characteristic shape can be observed:

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Figure 5: Demonstration of the leptokurtic price change profile

It should be noted that while outliers were excluded for improved chart readability, due to the extreme values skewing the chart dimensions, the leptokurtic profile is clearly visible. The data used to produce this figure (outliners included) demonstrates a kurtosis of around 7700, well above the minimum of 3 required for a distribution to be considered leptokurtic.

The slight skew to the positive is due to the general upward trend of stock prices overtime.

# Comparison: Probability of Return

Another potential study of the PDFs P(Z) and P(U) is to study the “probability of return” to the origin. This will give us a handle on the noise effecting each system, as a high “probability of return” will suggest lower noise. Here the data was taken from the Mantegna and Stanley paper (Stanley, Stock market dynamics and turbulence: Parallel analysis of fluctuation phenomena, 1997).

Explicitly we are studying the probability that after Δt the values for Y and V are the same. This is the same as p(Z=0) & p(U=0) after Δt.

This study is conducted on both data sets and also compared to a perfectly random gaussian process, here notated as Pg(0), the hypothetical gaussian process is:

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Figure 6: The probability of return for the economic data, P(Z=0) noted as circles, Pg(Z=0) noted as filled squares and the dotted line as a trendline superimposed on P(Z=0)

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Figure 7: The probability of return for the fluid data, P(U=0) noted as circles, Pg(0) noted as filled squares and the dotted line as a trendline superimposed on P(U=0)

In both Figure 6 & Figure 7 there is clear evidence that both the economic and fluid data do not match the gaussian process, as neither match the Pg(0) process. This point as also pointed out by Figure 5. Figure 6 shows a curve compatible with a levy stable process, suggesting like discussed in sections3.3 &4 that the financial data follows a random walk, with fat tails. Figure 7, the fluid data again doesn’t show a random walk, as it doesn’t fit its trendline, again there is a higher probability of return in both short and longer term windows. This is likely because of viscous forces in the short term and the fact that the fluid data will tend to return to its original value in the short term, unlike a random walk (that doesn’t have such a bias). Also the fluids shows a slightly smaller tendency for extreme changes.

# Conclusion

As discussed in the opening section of this paper, while there are some qualitative similarities between turbulence in the stock market and fluid turbulence, quantitatively some the basic statistical techniques suggest that stock market turbulence is random while fluid turbulence to the same techniques doesn’t seem random. It should be noted that this is only talking about when examined using the techniques applied in this paper.

Section 3.3 shows that while the fluid data confirms the predictions made my Kolmogorov, the financial data appears similar to a random walk.

Section 5 again shows that the financial data appears close to a random walk, where as the fluid data in short and longer intervals is more likely to return to the original values. This is likely because of the in the short term, there are small scale viscous forces at work and in the longer term the average fluid speed is fixed, whereas the financial data is constantly raising along an average trend.

But examining the charts in sections 3.1 & 3.2 can quite quickly demonstrate reasons for the differences between the two data sets in the examinations just discussed, the financial data set has both large jumps and an overall trend, unlike the fluid date.

In conclusion, while there are qualitative differences between fluid and stock market turbulence, on a quantitative level, this paper finds that there are less similarities at a close examination. This isn’t to say that there aren’t similarities however more examination of this subject is needed to find these if they exist.

# Bibliography

Cheng, Y. (2017, April 18). *Failure of Taylor's hypothesis in the atmospheric surface layer and its correction for eddy-covariance measurements.* Retrieved from Advancing Earth And Space Science: https://agupubs.onlinelibrary.wiley.com/doi/10.1002/2017GL073499#:~:text=Taylors'%20frozen%20turbulence%20hypothesis%20suggests,a%20single%20point%20in%20space.

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